Math 1B Midterm 1 Review Answers

[1] [a] By Net Change Theorem,

$$s(13)-s(1)=\int_1^{13} s'(t)\,dt=\int_1^{13} v(t)\,dt \quad\Rightarrow\quad s(13)=s(1)+\int_1^{13} v(t)\,dt \quad = \quad 21+\int_1^{13} v(t)\,dt \text{ feet}$$

[b] [i] $21+(12+18+17)*4=209$ feet

[ii] $21+(14+24+8)*4=205$ feet

[iii] $21+(18+17+5)*4=181$ feet

[2] $\int_{\pi}^{\frac{3\pi}{2}} 2\sin x\,dx=-2$ (if you used $\Delta x=\frac{\pi}{2n}$) or $\int_{2\pi}^{3\pi}\sin\frac{1}{2}x\,dx=-2$ (if you used $\Delta x=\frac{\pi}{n}$)

[3]

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{3}{n}\left[\left(-1+\frac{3i}{n}\right)^2+3\left(-1+\frac{3i}{n}\right)+2\right]$$

$$=\lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left(\frac{3i}{n}+\frac{9i^2}{n^2}\right)$$

$$=\lim_{n\to\infty}\frac{3}{n}\left(\frac{3}{n}\sum_{i=1}^{n} i+\frac{9}{n^2}\sum_{i=1}^{n} i^2\right)$$

$$=\lim_{n\to\infty}\frac{3}{n}\left(\frac{3}{n}\frac{n(n+1)}{2}+\frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right)$$

$$=\lim_{n\to\infty}3\left(\frac{3(n+1)}{2n}+\frac{3(n+1)(2n+1)}{2n^2}\right)$$

$$=3\left(\frac{3}{2}+\frac{6}{2}\right)$$

$$=\frac{27}{2}$$

[4]

$$\int_{-2}^{8} x\,dx-\int_{-2}^{8}\sqrt{25-(x-3)^2}\,dx$$

= area of 8×8 triangle – area of 2×2 triangle – area of semicircle of radius 5 (with center at $(3,0)$)

$$=\frac{1}{2}(8\times 8)-\frac{1}{2}(2\times 2)-\frac{1}{2}\pi(5)^2 \quad = \quad 30-\frac{25}{2}\pi$$

[5]

$$\frac{1}{2}\ \le\ \sin x\ \le\ 1 \text{ on } \left[\frac{\pi}{6},\frac{\pi}{2}\right]$$

$$\Rightarrow \frac{1}{2}x\ \le\ x\sin x\ \le\ x \text{ on } \left[\frac{\pi}{6},\frac{\pi}{2}\right]$$

$$\Rightarrow \int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\frac{1}{2}x\,dx\ \le\ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\sin x\,dx\ \le\ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\,dx$$

$$\Rightarrow \frac{1}{4}x^2\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}\ \le\ \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x\sin x\,dx\ \le\ \frac{1}{2}x^2\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$\Rightarrow \frac{1}{4}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{6}\right)^2\right] \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \le \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{6}\right)^2\right]$$

$$\Rightarrow \frac{1}{4}\left(\frac{\pi}{6}\right)^2 (3^2 - 1^2) \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \le \frac{1}{2}\left(\frac{\pi}{6}\right)^2 (3^2 - 1^2)$$

$$\Rightarrow \frac{1}{4}\frac{\pi^2}{36}(8) \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \le \frac{1}{2}\frac{\pi^2}{36}(8) \qquad \Rightarrow \frac{\pi^2}{18} \le \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} x \sin x \, dx \le \frac{\pi^2}{9}$$

[6] $\int_{7}^{-5} f(x)\,dx + \int_{-1}^{10} f(x)\,dx + \int_{10}^{7} f(x)\,dx = \int_{7}^{-5} f(x)\,dx + \int_{-1}^{7} f(x)\,dx = \int_{-1}^{-5} f(x)\,dx$

[7] [a] $g(3) = \int_{-2}^{3} f(t)\,dt = \frac{1}{2}(3\times 3) - \frac{1}{2}(2\times 1) = \frac{7}{2}$ and $g'(3) = f(3) = -1$

[b] Critical numbers of g occur where $g'(x) = 0$ or is undefined in the domain of g.
By FTC Part 1, since f is continuous on $[-5, 5]$, g is differentiable on $(-5, 5)$, and continuous on $[-5, 5]$.
So the domain of g includes $[-5, 5]$.
So, the critical numbers of g occur where $g'(x) = f(x) = 0$ or is undefined in $[-5, 5]$ ie. at $x = -4$, -2, 1 and 4.

[c] g is decreasing where $g'(x) = f(x) < 0$, so g is decreasing on $[-\infty, -4]$ and $[1, 4]$.

[d] Local minima of g occur at critical numbers of g where $g'(x) = f(x)$ changes from negative to positive,
so the local minima of g are $x = -4$ and 4.

[e] g is concave up where $g'(x) = f(x)$ is increasing, so g is concave up on $[-\infty, -3]$, $[-2, 0]$ and $[3, \infty]$.

[f] Inflection points of g occur where g is continuous and $g'(x) = f(x)$ changes from increasing to decreasing,
or vice versa, so the inflection points of g are $x = -3$, -2, 0 and 3.

[8] $g'(x) = 3x^2 \ln(1+x^6) - 2x\ln(1+x^4)$

$$g''(x) = 6x\ln(1+x^6) + 3x^2\frac{6x^5}{1+x^6} - \left(2\ln(1+x^4) + 2x\frac{4x^3}{1+x^4}\right)$$

$$g''(1) = 6\ln 2 + 3\left(\frac{6}{2}\right) - 2\ln 2 - 2\left(\frac{4}{2}\right) = 4\ln 2 + 5$$

[9]

$$\frac{d}{dx}\left(4 + \int_a^x \frac{1}{f(t)}\,dt\right) = \frac{d}{dx}2\sqrt{x}$$

$$\Rightarrow \frac{1}{f(x)} = \frac{1}{\sqrt{x}}$$

$$\Rightarrow f(x) = \sqrt{x}$$

$$4 + \int_a^x \frac{1}{\sqrt{t}}\,dt = 2\sqrt{x}$$

$$\Rightarrow 4 + 2\sqrt{t}\Big|_a^x = 2\sqrt{x}$$

$$\Rightarrow 4 + 2\sqrt{x} - 2\sqrt{a} = 2\sqrt{x}$$

$$\Rightarrow 4 - 2\sqrt{a} = 0$$

$$\Rightarrow a = 4$$

[10] the number of pounds Morgan gained from when he was 8 years old to when he was 15 years old

[11] [a] $v(t)-v(0)=\int_0^t v'(x)\,dx=\int_0^t a(x)\,dx \quad \Rightarrow \quad v(t)=v(0)+\int_0^t a(x)\,dx \quad = \quad 4+(3x-x^2)\Big|_0^t \quad = \quad 4+3t-t^2$

$$\int_1^6 v(t)\,dt \quad = \quad \left(4t+\frac{3}{2}t^2-\frac{1}{3}t^3\right)\Bigg|_1^6 \quad = \quad 4(6-1)+\frac{3}{2}(6^2-1^2)-\frac{1}{3}(6^3-1^3) \quad = \quad 20+\frac{105}{2}-\frac{215}{3}$$

$$= \quad 20+52\frac{1}{2}-71\frac{2}{3} \quad = \quad \frac{5}{6} \text{ meters}$$

[b] $v(t)=4+3t-t^2=(4-t)(1+t)\geq 0$ only on $[-1, 4]$

$$\int_1^6 |\,v(t)\,|\,dt \quad = \quad \int_1^4 (4+3t-t^2)\,dt+\int_4^6 -(4+3t-t^2)\,dt \quad = \quad \left(4t+\frac{3}{2}t^2-\frac{1}{3}t^3\right)\Bigg|_1^4 + -\left(4t+\frac{3}{2}t^2-\frac{1}{3}t^3\right)\Bigg|_4^6$$

$$= \quad 4(4-1)+\frac{3}{2}(4^2-1^2)-\frac{1}{3}(4^3-1^3)-\left(4(6-4)+\frac{3}{2}(6^2-4^2)-\frac{1}{3}(6^3-4^3)\right)$$

$$= \quad 12+\frac{45}{2}-21-\left(8+30-\frac{152}{3}\right) \quad = \quad 12+22\frac{1}{2}-21-8-30+50\frac{2}{3} \quad = \quad 26\frac{1}{6} \text{ meters}$$

[12] $\int_{-1}^2 (6+4f(2t+1))\,dt \quad = \quad \int_{-1}^2 6\,dt+4\int_{-1}^2 f(2t+1)\,dt$

Let $u=2t+1$

$\frac{du}{dt}=2 \quad \Rightarrow \quad dt=\frac{1}{2}du$

$t=-1 \quad \Rightarrow \quad u=-1$

$t=2 \quad \Rightarrow \quad u=5$

$$\int_{-1}^2 f(2t+1)\,dt \quad = \quad \int_{-1}^5 \frac{1}{2}f(u)\,du \quad = \quad \frac{1}{2}\int_{-1}^5 f(u)\,du \quad = \quad \frac{7}{2}$$

$$\int_{-1}^2 6\,dt+4\int_{-1}^2 f(2t+1)\,dt \quad = \quad 6(2-(-1))+4(\frac{7}{2}) \quad = \quad 32$$

[13] $\int_{-6}^6 (x^3\sqrt{2+\cos x}-\sqrt{144-(x+6)^2})\,dx \quad = \quad \int_{-6}^6 x^3\sqrt{2+\cos x}\,dx-\int_{-6}^6 \sqrt{144-(x+6)^2}\,dx$

$(-x)^3\sqrt{2+\cos(-x)}=-x^3\sqrt{2+\cos x} \quad \Rightarrow$ first integrand is odd and continuous, over a symmetric interval

$$\int_{-6}^6 x^3\sqrt{2+\cos x}\,dx \quad = \quad 0$$

$\int_{-6}^6 \sqrt{144-(x+6)^2}\,dx \quad =$ area of quarter circle of radius 12 (with center at $(-6, 0)$) $\quad = \quad \frac{1}{4}\pi(12)^2 \quad = \quad 36\pi$

$$\int_{-6}^6 (x^3\sqrt{2+\cos x}-\sqrt{144-(x+6)^2})\,dx \quad = \quad -36\pi$$